

BRILL-NOETHER-PETRI WITHOUT DEGENERATIONS

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Introduction

The purpose of this note is to show that curves generating the Picard group of a K3 surface X with $\text{Pic}(X) = \mathbf{Z}$ behave generically from the point of view of Brill-Noether theory. In particular, one gets a quick new proof of Gieseker's theorem [5] concerning the varieties of special divisors on a general algebraic curve.

Let C be a smooth irreducible complex projective curve of genus g . One says that C satisfies *Petri's condition* if the map

$$\mu_0: H^0(A) \otimes H^0(\omega_C \otimes A^*) \rightarrow H^0(\omega_C)$$

defined by multiplication is injective for every line bundle A on C . Roughly speaking, this condition means that the varieties $W_d^r(C)$ of special divisors on C have the properties one would naively expect. Specifically, it implies that $W_d^r(C)$ is smooth away from $W_d^{r+1}(C)$, and that $W_d^r(C)$ (when nonempty) has the postulated dimension $\rho(r, d, g) =_{\text{def}} g - (r + 1) \cdot (g - d + r)$. We refer to [1] for the definition of $W_d^r(C)$, and for a detailed discussion of Petri's condition and its role in Brill-Noether theory. One of the most basic results of this theory is Gieseker's theorem [5] that Petri's condition does in fact hold for the generic curve of genus g .

We prove here the following

Theorem. *Let X be a complex projective K3 surface, and let $C_0 \subset X$ be a smooth connected curve. Assume that every divisor in the linear system $|C_0|$ is reduced and irreducible. Then the general curve $C \in |C_0|$ satisfies Petri's condition.*

The hypothesis is satisfied in particular when $\text{Pic}(X)$ is infinite cyclic, generated by the class of C_0 . But for any integer $g \geq 2$ there exists a $K3$ surface X with $\text{Pic}(X) = \mathbf{Z} \cdot [C_0]$ for some curve C_0 of genus g , and thus the theorem implies Gieseker's result [5].

The study of special divisors on a general curve has traditionally centered around degeneration arguments. One of the first results in this area was due to Griffiths and Harris [7], who proved the assertion of Brill and Noether that if C is a general curve of genus g , then $\dim W_d^r(C) = \rho(r, d, g)$ provided that $\rho(r, d, g) \geq 0$. Their method was to deduce the theorem from an analogous statement for a rational curve with g nodes, which in turn was proved by a further degeneration. To prove Petri's conjecture, Gieseker [5] combined some rather elaborate combinatorial arguments with a systematic analysis of the limiting linear series on reducible curves arising in a degeneration of g -nodal \mathbf{P}^1 's. Eisenbud and Harris [2] subsequently streamlined Gieseker's proof by using a different degeneration, and they have recently extended and given several interesting new applications of these techniques (cf. [4]).

By contrast, the proof of the theorem here does not require any degenerations. Instead the method is simply to exhibit smooth families of g_d^r 's. Specifically, we consider triples (C, A, τ) consisting of a nonsingular curve $C \subset X$ in the linear system $|C_0|$, a line bundle $A \in W_d^r(C)$ such that both A and $\omega_C \otimes A^*$ are base-point free, and an isomorphism τ mod scalars of $H^0(A)$ with a fixed vector space of dimension $r + 1$. Such triples are parametrized by a variety P_d^r , and one has an evident map $\pi: P_d^r \rightarrow |C_0|$. The tangent spaces to P_d^r and the derivative of π are computed cohomologically in terms of certain vector bundles $F_{C,A}$ on X which we study in §1. One finds in particular that these bundles have only trivial endomorphisms so long as $|C_0|$ does not contain any reducible curves. Much as in [10] this allows us to show in §2 that P_d^r is nonsingular, and that moreover the morphism π is smooth at (C, A, τ) if and only if the Petri μ_0 map for A is injective. The theorem then follows (§3) from the generic smoothness of π . In as much as it avoids the combinatorics involved in degenerational proofs, the present approach to Brill-Noether-Petri would seem to be simpler than the traditional one. On the other hand, as in [2] the argument only works in characteristic zero, and these techniques do not yield the theorem of Kempf [8] and Kleiman-Laksov [9] that $W_d^r(C)$ is nonempty when $\rho(r, d, g) \geq 0$ (which however is elementary nowadays; cf. [1, Chapter VII]).

Special divisors on a curve C on a $K3$ surface X appear to have been first considered by Reid [13], who showed that under suitable numerical hypotheses a special pencil on C is the restriction of one on X . A beautiful conjecture of Mumford, Harris and Green (see [6, §5]) asserts that all curves in a given linear

series on X have the same Clifford index. This conjecture—which would generalize the well-known fact that if $C_0 \subset X$ is hyperelliptic, then so too is any other smooth curve in $|C_0|$ —has been verified in special cases by Donagi and Morrison, and by Green and the author. Serrano-Garcia [14] has extended some of Reid’s results to surfaces other than $K3$ ’s.

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1. The vector bundles $F_{C,A}$

This section is devoted to the study of certain vector bundles that play an important role in the argument. But first some notation. Throughout the paper X denotes a complex projective $K3$ surface, and $C_0 \subset X$ is a smooth irreducible curve of genus g . Given a curve C , and integers d and r , we define

$$V_d^r(C) \subset \text{Pic}^d(C)$$

to be the open subset of $W_d^r(C)$ consisting of line bundles A on C such that:

- (i) $h^0(A) = r + 1$, $\text{deg}(A) = d$; and
- (ii) both A and $\omega_C \otimes A^*$ are generated by their global sections.

Fix now a smooth curve $C \subset X$ in the linear series $|C_0|$, and a line bundle $A \in V_d^r(C)$. We associate to the pair (C, A) a vector bundle $F_{C,A}$ on X , of rank $r + 1$, as follows. Thinking of A as a sheaf on X , there is a canonical surjective evaluation map

$$e_{C,A}: H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow A$$

of \mathcal{O}_X -modules. Take

$$F_{C,A} \stackrel{\text{def}}{=} \ker e_{C,A}$$

to be its kernel. [Note that A , being locally isomorphic to \mathcal{O}_C , has homological dimension 1 over \mathcal{O}_X . Hence $F_{C,A}$ is indeed a vector bundle.]

The basic properties of these bundles are easily determined. Specifically, setting $F = F_{C,A}$ one has by construction the exact sequence

$$(1.1) \quad 0 \rightarrow F \rightarrow H^0(A) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow A \rightarrow 0$$

of sheaves on X . Since $\mathcal{O}_X = \mathcal{O}_X$, dualizing (1.1) gives:

$$(1.2) \quad 0 \rightarrow H^0(A)^* \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow F^* \rightarrow \omega_C \otimes A^* \rightarrow 0,$$

and from (1.1) and (1.2) one sees that:

- (i) $c_1(F) = -[C_0]$, $c_2(F) = \deg(A) = d$;
- (ii) F^* is generated by its global sections [recall: $h^1(\mathcal{O}_X) = 0$];
- (iii) $H^0(F) = H^2(F^*) = 0$,
 $H^1(F) = H^1(F^*) = 0$,
 $h^0(F^*) = h^0(A) + h^1(A)$.

Furthermore, one has:

- (iv) $\chi(F \otimes F^*) = 2 \cdot h^0(F \otimes F^*) - h^1(F \otimes F^*) = 2 - 2 \cdot \rho(A)$,
 where $\rho(A) = g(C) - h^0(A) \cdot h^1(A)$.

Proof. The first equality follows from Serre duality. If E is a vector bundle of rank e on X , Riemann-Roch gives $\chi(E \otimes E^*) = (e - 1) \cdot c_1(E)^2 - 2e \cdot c_2(E) + 2e^2$. Now compute.

The presence or absence of reducible curves in $|C_0|$ comes into play via

Lemma 1.3. *Fix a smooth curve C in $|C_0|$ and a line bundle $A \in V_d^r(C)$, and let $F = F_{C,A}$. If F has nontrivial endomorphisms, i.e. if $h^0(F \otimes F^*) \geq 2$, then the linear system $|C_0|$ contains a reducible (or multiple) curve.*

Proof. Set $E = F^*$. Since $h^0(E \otimes E^*) \geq 2$, there exists by a standard argument a nonzero endomorphism $v: E \rightarrow E$ which drops rank everywhere on X . [Take any endomorphism w of E , $w \neq (\text{const}) \cdot 1$, and set $v = w - \lambda \cdot 1$, where λ is an eigenvalue of $w(x)$ for some $x \in X$. Then

$$\det(v) \in H^0(\det(E^*) \otimes \det(E)) = H^0(\mathcal{O}_X)$$

vanishes at x , and hence is identically zero.] Let

$$N = \text{im } v, \quad M_0 = \text{coker } v,$$

and put

$$M = M_0/T(M_0),$$

where $T(M_0)$ is the torsion subsheaf of M_0 . Thus

$$[C_0] = c_1(E) = c_1(N) + c_1(M) + c_1(T(M_0))$$

in the Chow group $A_1(X) = \text{Pic}(X)$. Now $c_1(T(M_0))$ is represented by a nonnegative linear combination of the codimension one irreducible components (if any) of $\text{supp}(T(M_0))$. So it is enough to show that $c_1(N)$ and $c_1(M)$ are represented by nonzero effective curves. But N and M are torsion-free sheaves of positive rank, and—being quotients of E —are generated by their global sections. Furthermore, since $H^0(E^*) = 0$ neither of these can be trivial vector bundles. So the lemma follows from the elementary fact:

Let U be a torsion-free sheaf on a smooth projective surface. If U is generated by its global sections, then $c_1(U)$ is represented by an effective (or zero) divisor. Moreover $c_1(U) = 0 \Leftrightarrow U$ is a trivial vector bundle.

Indeed, the double dual U^{**} of U is locally free, and the canonical inclusion $U \rightarrow U^{**}$ is an isomorphism outside of a finite set (cf. [12, II.1.1]). Thus $c_1(U) = c_1(U^{**})$, and U^{**} is generated by its sections away from finitely many points. Therefore $H^0(\det(U^{**})) \neq 0$, and (by Porteous) $c_1(U^{**}) = 0$ if and only if U^{**} —and hence also U —is a trivial bundle. q.e.d.

It is amusing to note that the lemma already yields a special case of the Brill-Noether theorem [7], namely that a general curve C of genus g does not carry any line bundle A with $\rho(A) [= g(C) - h^0(A) \cdot h^1(A)] < 0$. In fact:

Corollary 1.4. *Assume that every member of the linear series $|C_0|$ is reduced and irreducible. Then for every smooth curve $C \in |C_0|$ and every line bundle A on C one has $\rho(A) \geq 0$.*

When $h^0(A) = 2$ the corollary was proved by Donagi and Morrison (unpublished) using very different methods of Reid [13], and independently by Reid himself (private communication). Compare also [3].

Proof of Corollary 1.4. Observe that if B is a base-point free special line bundle on C , and if Δ is the divisor of base-points of $\omega_C \otimes B^*$, then $B(\Delta)$ is again base-point free. Hence we can assume in (1.4) that both A and $\omega_C \otimes A^*$ are generated by their global sections, and then the assertion follows from (iv) and (1.3).

2. Infinitesimal calculations

Keeping notation as in §1, we now fix positive integers r and d , and a vector space H of dimension $r + 1$.

Definition 2.1. *Let P'_d denote the quasi-projective scheme (constructed below) parametrizing the set of all triples (C, A, λ) , where:*

- (i) $C \subset X$ is a smooth curve in the linear system $|C_0|$;
- (ii) $A \in V'_d(C)$; and
- (iii) λ is a surjective homomorphism of \mathcal{O}_X -modules:

$$\lambda: H \otimes_C \mathcal{O}_X \rightarrow A \rightarrow 0$$

inducing an isomorphism $H \cong H^0(A)$, two such homomorphisms being identified if they differ only by multiplication by a nonzero scalar.

Construction of P'_d : P'_d is an open subset of a Hilbert scheme classifying curves in $X \times \mathbf{P}(H)$. Specifically, given a triple (C, A, λ) as above, the quotient $\lambda|_C: H \otimes_C \mathcal{O}_C \rightarrow A$ determines an embedding

$$C \subset \mathbf{P}(H \otimes_C \mathcal{O}_X) = X \times \mathbf{P}(H),$$

and distinct triples give rise to distinct subvarieties of $X \times \mathbf{P}(H)$. The subschemes of $X \times \mathbf{P}(H)$ arising in this manner are parametrized by a Zariski-open subset of the Hilbert scheme of curves in $X \times \mathbf{P}(H)$ (with appropriate Hilbert

polynomial defined with respect to some ample divisor on $X \times \mathbf{P}(H)$). We take this open set to be P'_d .

Observe that there is a natural morphism

$$\pi: P'_d \rightarrow |C_0|$$

sending a triple (C, A, λ) to the point $\{C\}$. Note also that for every $(C, A, \lambda) \in P'_d$, the sheaf $\ker \lambda$ is isomorphic to the bundle $F_{C,A}$ introduced in §1. Consequently the discussion of §1 applies to these kernels.

The basic fact for us is that one has good infinitesimal control over P'_d and π :

Proposition 2.2. *Fix any point $(C, A, \lambda) \in P'_d$, and let $F = \ker \lambda$. Assume that $h^0(F \otimes F^*) = 1$. Then:*

- (i) P'_d is smooth at (C, A, λ) , of dimension $\rho(A) + g + \{h^0(A)^2 - 1\}$; and
- (ii) The map π is smooth at (C, A, λ) , i.e. $d\pi_{(C,A,\lambda)}$ is surjective, if and only if the Petri homomorphism

$$\mu_0: H^0(A) \otimes H^0(\omega_C \otimes A^*) \rightarrow H^0(\omega_C)$$

is injective.

Remark. Observe that there is no assumption on the integers r and d . However it may well be that P'_d is empty [cf. Corollary 1.4].

Proof of Proposition 2.2. Consider the embedding $C \subset X \times \mathbf{P}(H)$ determined by λ . Denoting by $\Phi: C \rightarrow \mathbf{P}(H)$ the projection of C to $\mathbf{P}(H)$, one has a canonical exact sequence of tangent and normal bundles:

$$(*) \quad 0 \rightarrow \Phi^*(\Theta_{\mathbf{P}(H)}) \rightarrow N_{C/X \times \mathbf{P}(H)} \rightarrow N_{C/X} \rightarrow 0,$$

and $d\pi_{(C,A,\lambda)}$ is identified with the resulting homomorphism

$$T_{(C,A,\lambda)}P'_d = H^0(N_{C/X \times \mathbf{P}(H)}) \rightarrow H^0(N_{C/X}) = T_{(C)}|C_0|.$$

Grant for the time being the following

Claim. If $h^0(F \otimes F^*) = 1$, then the map

$$(**) \quad H^1(N_{C/X \times \mathbf{P}(H)}) \rightarrow H^1(N_{C/X})$$

determined by (*) is bijective.

Then first of all one gets an isomorphism $\text{coker } d\pi_{(C,A,\lambda)} \cong H^1(\Phi^*(\Theta_{\mathbf{P}(H)}))$. But $\Phi = \Phi_A$ is the morphism determined by the complete linear system associated to A , and hence $H^1(\Phi^*(\Theta_{\mathbf{P}(H)}))$ is Serre dual to $\ker \mu_0$. This proves (ii).

For (i) we argue much as in [10] that the obstructions to the smoothness of the Hilbert scheme of $X \times \mathbf{P}(H)$ at (C, A, λ) vanish. Specifically, let R be a local artinian \mathbf{C} -algebra, let $I \subset R$ be a one-dimensional square-zero ideal, and set $S = R/I$. Consider an infinitesimal deformation

$$(+ \quad) \quad \underline{C} \subset X \times \mathbf{P}(H) \times \text{Spec}(S)$$

of C in $X \times \mathbf{P}(H)$ over $\text{Spec}(S)$. The obstruction to extending $(+)$ to a deformation over $\text{Spec}(R)$ is given by an element $o_{(+)} \in H^1(N_{C/X \times \mathbf{P}(H)})$. On the other hand, $(+)$ determines by projection an infinitesimal deformation

$$(\#) \quad \underline{C} \subset X \times \text{Spec}(S)$$

of C in X , and one has a corresponding obstruction class $o_{(\#)} \in H^1(N_{C/X})$. Furthermore, $o_{(+)}$ maps to $o_{(\#)}$ under the homomorphism $(**)$; this can be checked, e.g., using the explicit description of the obstruction classes in [11, Lecture 23] by observing that the local equation of \underline{C} in $X \times \mathbf{P}(H) \times \text{Spec}(S)$ can be taken as one of the equations locally cutting out \underline{C} in $X \times \mathbf{P}(H) \times \text{Spec}(S)$. But the Hilbert scheme $|C_0|$ of C in X is smooth, and hence $o_{(\#)} = 0$. Therefore $o_{(+)} = 0$ thanks to the claim, and this proves that P'_d is smooth at (C, A, λ) . (One could also deduce (i) from Theorem (0.1) of [10].)

It remains to verify the claim. Denoting by p and q the projections of $X \times \mathbf{P}(H)$ onto X and $\mathbf{P}(H)$ respectively, note first that C is defined in $X \times \mathbf{P}(H)$ as the zero-locus of the evident section of $p^*(F^*) \otimes q^*(\mathcal{O}_{\mathbf{P}(H)}(1))$. Therefore

$$N_{C/X \times \mathbf{P}(H)} = F^*|C \otimes A.$$

We next compute $h^1(C, F^*|C \otimes A) = h^1(X, F^* \otimes A)$. To this end, observe that since F^* is locally free, λ determines an exact sequence

$$0 \rightarrow F \otimes F^* \rightarrow H \otimes_C F^* \rightarrow A \otimes F^* \rightarrow 0$$

of sheaves on X . Using the computations of $H^i(F^*)$ in §1 one sees that $H^1(X, A \otimes F^*) = H^2(X, F \otimes F^*)$, and so by duality plus the hypothesis on $F \otimes F^*$ one finds that $h^1(N_{C/X \times \mathbf{P}(H)}) = 1$. Since also $h^1(N_{C/X}) = h^1(\omega_C) = 1$, the claim follows. Finally, using facts (iii) and (iv) from §1, one gets the stated value for $h^0(X, F^* \otimes A) = \dim_{(C,A,\lambda)} P'_d$.

Remark. Suppose that the linear system $|C_0|$ does not contain any reducible members. Then it follows from the proposition and Lemma 1.3 that P'_d (if nonempty) has pure dimension $g + \rho(d, r, g) + \{(r + 1)^2 - 1\}$. Observing that the fiber of π over a point $\{C\} \in |C_0|$ is a $\text{PGL}(r + 1)$ -bundle over $V'_d(C)$, one can use this to give a proof of the Brill-Noether theorem of Griffiths and Harris [7]. But at this point it is quicker for us to get dimensionality via Petri.

3. Proof of the Theorem

We assume that the linear system $|C_0|$ does not contain any reducible or multiple members, and we wish to show that almost every curve in $|C_0|$ satisfies Petri's condition.

To begin with fix arbitrary positive integers r and d . We claim that there is a nonempty Zariski-open set $U'_d \subset |C_0|$ of smooth curves such that for all $C \in U'_d$:

$$\mu_0: H^0(A) \otimes H^0(\omega_C \otimes A^*) \rightarrow H^0(\omega_C) \text{ is injective} \\ \text{for every line bundle } A \in V'_d(C).$$

Indeed, it follows from Lemma 1.3 and the assumption on $|C_0|$ that for any point $(C, A, \lambda) \in P'_d$, the bundle $F = \ker \lambda$ satisfies $h^0(F \otimes F^*) = 1$. Thus by Proposition 2.2 the variety P'_d is nonsingular (or empty). As we are in characteristic zero the theorem on generic smoothness applies, and there exists a nonempty open set $U'_d \subset |C_0|$ over which the map $\pi: P'_d \rightarrow |C_0|$ is smooth. Invoking the proposition again, it follows that U'_d has the stated property.

We assert next that there is a nonempty open set $U \subset |C_0|$ of smooth curves such that for any $C \in U$:

$$\mu_0 \text{ is injective for every line bundle } A \text{ on } C \text{ such that both } A \\ \text{and } \omega_C \otimes A^* \text{ are generated by their global sections.}$$

In fact, for a fixed genus g the injectivity of μ_0 for A is nontrivial for only finitely many values of $d = \deg(A)$ and $r = r(A)$ [e.g., $0 \leq 2r \leq d \leq 2g - 2$]. It suffices to take U to be the intersection of the corresponding U'_d 's.

Using the remark at the beginning of the proof of Corollary 1.4, the theorem now follows from the observation that if D is any effective divisor on C , and if Δ is the divisor of base-points of $|D|$, then the injectivity of μ_0 for $\mathcal{O}_C(D - \Delta)$ implies the injectivity of μ_0 for $\mathcal{O}_C(D)$.

Remark. It is not generally the case that Petri's condition holds for *all* smooth curves in $|C_0|$. Furthermore, one cannot avoid the hypothesis on $|C_0|$: e.g. for $n \geq 2$ the general member of $|n \cdot C_0|$ does not satisfy Petri. Similarly one can not expect to weaken too greatly the hypothesis that X be a $K3$, since for instance the theorem already fails for the general surface of degree ≥ 5 in \mathbf{P}^3 .

References

- [1] E. Arbarello, M. Cornalba, P. Griffiths & J. Harris, *Geometry of algebraic curves*, Volume 1, Springer, Berlin, 1985.
- [2] D. Eisenbud & J. Harris, *A simpler proof of the Gieseker-Petri theorem on special divisors*, *Invent. Math.* **74** (1983) 269–280.
- [3] ———, *On the Brill-Noether theorem*, in *Algebraic Geometry—Open Problems*, Lecture Notes in Math., Vol. 997, Springer, Berlin, 1983, 131–137.
- [4] ———, *Limit linear series: basic theory*, to appear.

- [5] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982) 251–275.
- [6] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geometry **19** (1984) 125–171.
- [7] P. Griffiths & J. Harris, *The dimension of the variety of special linear systems on a general curve*, Duke Math. J. **47** (1980) 233–272.
- [8] G. Kempf, *Schubert methods with an application to algebraic curves*, Publ. Math. Centrum, Amsterdam, 1972.
- [9] S. Kleiman & D. Laksov, *On the existence of special divisors*, Amer. J. Math. **94** (1972) 431–436.
- [10] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984) 101–116.
- [11] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Math. Studies, No. 59, Princeton University Press, Princeton, NJ, 1966.
- [12] C. Okonek, M. Schneider & H. Spindler, *Vector bundles on complex projective spaces*, Birkhauser, Basel, 1980.
- [13] M. Reid, *Special linear systems on curves lying on a K-3 surface*, J. London Math. Soc. **13** (1976) 454–458.
- [14] F. Serrano-Garcia, *Surfaces having a hyperplane section with a special pencil*, Thesis (Brandeis), 1985.

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